Universal Error-Reducing Methodology on Option Pricing

Hua-Yi Lin

Advisor: Tian-Shyr Dai

November 29, 2011

Agenda

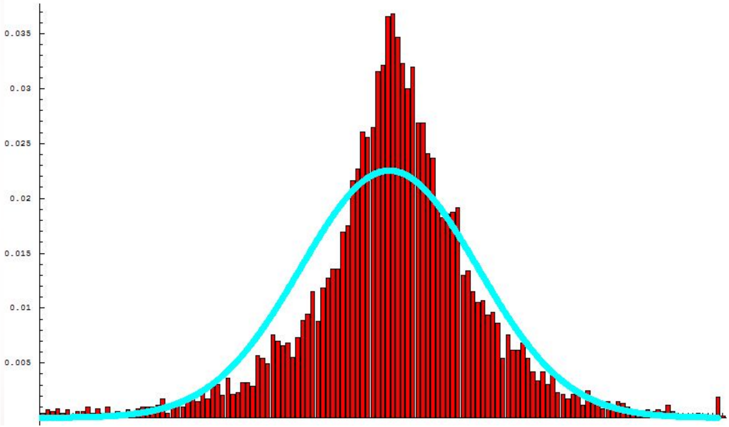
· Why we use Lévy processes for option pricing?

· What is a Lévy process?

· Option pricing with Lévy process – using FFT

· Our improvements on P.Carr and D.B.Madan (1999). Option valuation using the fast Fourier transform. *Journal of Computational Finance.*

Why we use Lévy processes for option pricing?



Recall

A stochastic process $X = \{X_t, t \geq 0\}$ is a *standard Brownian motion* on some probability space$(\Omega, \mathcal{F}, \mathbb{P})$ if

(i) $X_0 = 0$ a.s.

(ii) X has independent increments.(i.e. $X_{t_2} - X_{t_1} \coprod X_{t_4} - X_{t_3}$)

(iii) X has stationary increments.(i.e. $X_{t_1+s} - X_{t_1} \underline{\underline{d}} X_{t_2+s} - X_{t_2}$)

(iv) $X_{t+s} - X_t \sim N(0, s)$

(i.e. $X_{t+s} - X_t$ has characteristic function $\phi(u) = e^{-\frac{1}{2}s^2u^2}$)

If we generalize ϕ as

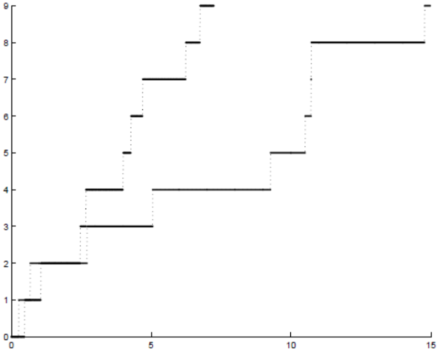
$$\phi(u) = e^{-\frac{1}{2}s^2u^2 + i\gamma u + \int_{-\infty}^{\infty} e^{iux} - 1 - iux\, 1_{\{|x| \leq 1\}}\, \nu(dx)}$$

then X defines a Lévy process.

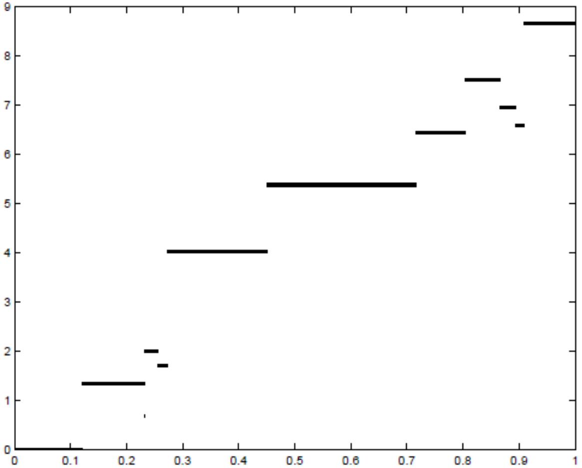
Example (Poisson process)

A Poisson process is a stochastic process such that the number of events in time interval$(t, t+s]$ follows a Poisson distribution with associated parameter λs.

(i.e. $\mathbb{P}(N(t+s) - N(t) = k) = \frac{e^{-\lambda s}(\lambda s)^k}{k!}$ $k = 0, 1, 2, ...$)

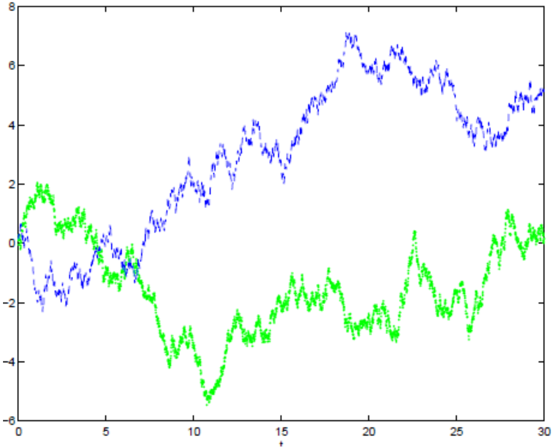


Example (Compound Poisson process(with normal jump size))



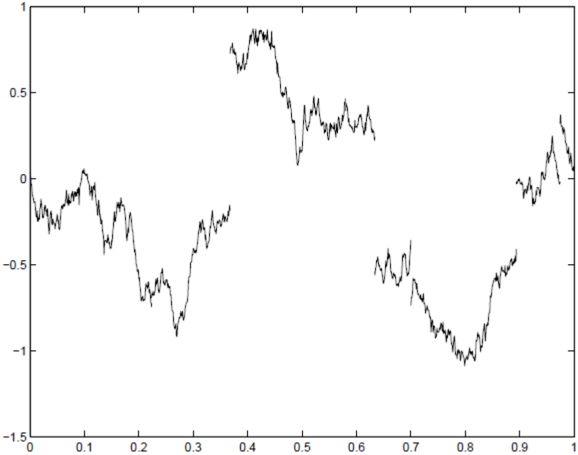
Example (standard Brownian motion)

$dlogS = \mu\, dt + \sigma\, dW$



What is a Lévy process?

Example (Merton's jump diffusion model)
$dlogS = \mu\, dt + \sigma\, dW + J\, dN, \quad J \sim N(\alpha, \beta)$



The FFT is an efficient algorithm for computing the sum

$w(k) = \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(k-1)}\chi(j) \quad \mathit{for}\ k = 1 \ldots N$

Carr and Madan(1999) shows that

$$C_T(k) = \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} e^{-ivk}\psi_T(v)\,dv = \frac{e^{-\alpha k}}{\pi}\int_{0}^{\infty} e^{-ivk}\psi_T(v)\,dv$$

where $\psi_T(v) = \frac{e^{-rT}\phi_T(v-(\alpha+1)i)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}$ is a function of a characteristic function.

$\therefore C_T(k) \approx \frac{e^{-\alpha k}}{\pi}\sum_{j=1}^{N} e^{-iv_j k}\psi_T(v_j)\eta$

$= \frac{e^{-\alpha k}}{\pi}\sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(u-1)} e^{ibv_j}\psi_T(v_j)\eta$

Our improvements on Carr and Madan (1999)

Our idea: $\psi_T(v) = \psi_T^{proxy}(v) + \psi_T^{residue}(v)$

$$\therefore C_T(k) = \frac{e^{-\alpha k}}{\pi} \int_0^\infty e^{-ivk} \psi_T(v)\, dv$$

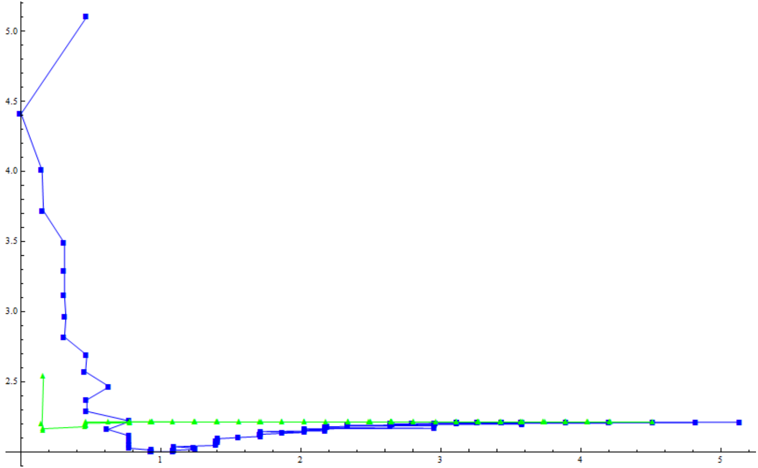
$$= \frac{e^{-\alpha k}}{\pi} \int_0^\infty e^{-ivk} [\psi_T^{proxy}(v) + \psi_T^{residue}(v)]\, dv$$

$$= \underbrace{C_T^{proxy}(k)}_{analytic\ formula} + \underbrace{\frac{e^{-\alpha k}}{\pi} \int_0^\infty e^{-ivk} \psi_T^{residue}(v)\, dv}_{FFT}$$

In Merton's jump-diffusion model,

$$C_T^{MJD}(k) = \sum_{j=0}^{\infty} \frac{e^{-\lambda' T}(\lambda' T)^j}{j!} C_T^{BS}(k)(S_0, T, \sigma_j, r_j) := C_T^{proxy}(k)$$

Example (FFT pricing on DE model with MJD proxy used)



Our improvements on Carr and Madan (1999)

Example (FFT pricing on VG model with MJD proxy used)